PLURALITYVETO: A Simple Voting Rule Achieving Optimal Metric Distortion

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Abstract

The metric distortion framework posits that n voters and m candidates are jointly embedded in a metric space such that voters rank candidates that are closer to them higher. A voting rule's purpose is to pick a candidate with minimum total distance to the voters, given only the rankings, but not the actual distances. As a result, in the worst case, each deterministic rule picks a candidate whose total distance is at least three times larger than that of an optimal one, i.e., has distortion at least 3. A recent breakthrough result showed that achieving this bound of 3 is possible; however, the proof is non-constructive, and the voting rule itself is a complicated exhaustive search.

Our main result is an extremely simple voting rule, called PLURALITYVETO, which achieves the same optimal distortion of 3. Each candidate starts with a score equal to his number of first-place votes. These scores are then gradually decreased via an *n*-round veto process in which a candidate drops out when his score reaches zero. One after the other, voters decrement the score of their bottom choice among the standing candidates, and the last standing candidate wins. We give a one-paragraph proof that this voting rule achieves distortion 3. This rule is also immensely practical, and it only makes two queries to each voter, so it has low communication overhead. We also show that a straightforward extension can be used to give a constructive proof of the more general Ranking-Matching Lemma of Gkatzelis et al.

We also generalize PLURALITYVETO into a class of randomized voting rules in the following way: PLURALITYVETO is run only for k < n rounds; then, a candidate is chosen with probability proportional to his residual score. This general rule interpolates between RANDOMDICTATORSHIP (for k = 0) and PLURALITYVETO (for k = n - 1), and k controls the variance of the output. We show that for all k, this rule has expected distortion at most 3.

1 Introduction

Voting is a fundamental process for reaching consensus and plays a vital role in democracies, organizations and businesses. Even honeybees use a type of voting to decide among potential nest sites; each bee casts a *numerical* vote indicating the intensity of its preferences, and the site with the highest score wins [29]. Quantifying preference intensity is not so easy for complicated problems that humans face; thus, the predominant approach is to elicit, from each voter, a preference *ranking* over candidates. This creates a need for a *voting rule* that determines the winner from these preferences. Numerous rules have been proposed over the years, with no consensus on a "best" rule.

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A key contribution of computer science in this regard has been viewing the ranking of a voter as a proxy for the latent cost she¹ incurs if some candidate wins. Then, an optimal candidate can be defined as one minimizing the total cost, and vote aggregation can be interpreted as an optimization problem with missing information. Due to the missing information, a voting rule can be thought of as an approximation algorithm, whose worst-case performance is referred to as its *distortion* in this setting.

Without any structures on the costs, not much can be achieved in terms of distortion [11]. An important contribution due to Anshelevich et al. [2] (see also the journal version [3] and recent surveys [5, 4]) was to assume that the n voters and m candidates are jointly embedded in a metric space, and voters rank candidates by increasing distance.² This viewpoint is motivated by observing that each candidate exhibits a standpoint on various issues that voters care about, and each voter also has a standpoint on these issues that is presumably reflected on her ballot. One would then expect that voters rank candidates whose standpoints are "closer" to theirs higher. Note that voting rules do not have access to this space; the only available information is the rankings, which serve as an *ordinal* proxy for the (cardinal) distances. The worst-case approximation specifically for metric costs is called *metric distortion*. (Formal definitions of all concepts are given in Section 2.)

The metric distortion framework has proved to be a fruitful analysis tool. In their initial work, Anshelevich et al. [2] established a lower bound of 3 on the distortion of any deterministic voting rule, and showed that the Copeland rule nearly matches the lower bound by achieving distortion 5. Several subsequent papers worked towards closing this gap. Initially, the Ranked Pairs rule was conjectured to achieve distortion 3. This was disproved by Goel et al. [18] who gave a lower bound of 5; Kempe [21] strengthened the lower bound to $\Omega(\sqrt{m})$. The first improvement over the upper bound of 5 was due to Munagala and Wang [24], who achieved distortion $2 + \sqrt{5} \approx 4.23$ using a novel asymmetric variant of the Copeland rule. The distortion-3 conjecture was recently resolved in a breakthrough result by Gkatzelis et al. [17], using a novel voting rule called PLURALITYMATCHING.

One of the main drawbacks of PLURALITYMATCHING is that it is unusually complex for a voting rule in the conventional sense. The winner is selected based on perfect matchings in certain bipartite graphs, which we will discuss shortly. Due to the complex nature of the voting rule, it is not even obvious that there always exists a winner in PLURALITYMATCHING; indeed, this existence proof was the main contribution of Gkatzelis et al. [17] over the prior work of Munagala and Wang [24] and Kempe [21]. The rule is also almost certainly too technical to be understood by the general public.

Our main contribution is an extremely simple voting rule, called PLURALITYVETO, which achieves the same optimal metric distortion of 3.

Under PLURALITYVETO, each candidate starts with a score equal to his *plurality score*, i.e., the number of first-place votes he receives. These scores are then gradually decreased; when the score of a candidate reaches zero, he is eliminated. Voters are processed one by one in an arbitrary order: when it is the turn of a voter, she decrements the score of her bottom choice among uneliminated candidates. Since the initial scores add up to the number of voters, all of the candidates will be eliminated at the end. The last eliminated candidate wins. Notice that this rule does not even require access to the voters' full rankings. Aside from the top choices, it only requires from each voter her bottom choice among uneliminated candidates. Thus, when implemented via sequential queries to voters, PLURALITYVETO also has low communication overhead. We summarize our main result as follows. (A formal presentation and proof are given in Section 3.)

¹For ease of presentation, we use female pronouns for voters and male pronouns for candidates throughout.

²This assumption generalizes the classic notion of *single-peaked preferences* [7, 23].

Contribution 1. PLURALITYVETO has the optimal metric distortion of 3 and can be implemented to require each voter to communicate only $O(\log m)$ bits to the voting rule.

A candidate with a strict majority of first-place votes wins under PLURALITYVETO, regardless of the order in which voters are processed. Thus, one can think of the elimination process (or, as we call it, *multi-round veto*) as a runoff stage. This makes our rule conceptually simple as well; it is just plurality voting followed by multi-round veto. In this respect, it resembles instant runoff voting, which is used in national elections in several countries. As in instant runoff voting, PLURALITYVETO can be used by eliciting from each voter her full ranking, so that the runoff stage can be run instantly. Alternatively, it can be arranged as a two-stage election in which voters first cast a vote for their top choice, and in the following stage, each voter cancels the vote of another voter.

PLURALITYMATCHING and its analysis are based on *domination graphs*, a family of bipartite graphs G(c) (one per candidate c) between voters and voters defined in Section 2. Munagala and Wang [24] and Kempe [21] had shown that if G(c) has a perfect matching, then c has distortion at most 3. The key contribution of Gkatzelis et al. [17] was to show that such a candidate c always exists. In fact, they proved a generalization, called the Ranking-Matching Lemma, which shows the existence of a graph with a *weighted* perfect bipartite matching in a more general class of bipartite graphs. We also give (in Section 4) a constructive one-paragraph proof of this more general Ranking-Matching Lemma, using a variant FRACTIONALVETO of PLURALITYVETO which decreases weights fractionally, rather than integrally, in each step.

Finally, we generalize PLURALITYVETO to a class of randomized voting rules that choose a candidate with probability proportional to his residual score at the k^{th} round of PLURALITYVETO, which we refer to as k-ROUNDPLURALITYVETO. When k = 0, this more general rule specializes to the well-known rule RANDOMDICTATORSHIP, which chooses the top choice of a uniformly random voter. Hence, randomizing the outcome proportional to the scores achieves distortion 3-2/n when k = 0, as shown in [1]. In Section 5, we show that k-ROUNDPLURALITYVETO in fact achieves distortion at most 3 for *all* k. In that way, it interpolates between RANDOMDICTATORSHIP (for k = 0) and PLURALITYVETO (for k = n - 1), and the parameter k controls the variance of the output, but does not affect the distortion. The intuition behind this is that PLURALITYVETO repeatedly decreases scores for "extreme" candidates, and thus is likely to end up with central ones. We elaborate on this intuition in the context of Peer Selection in Section 6.

Contribution 2. *k*-ROUNDPLURALITYVETO has expected metric distortion at most 3 for any number of rounds k.

This upper bound is almost tight, since any rule that can only elect candidates who are the top choice of at least one voter must have distortion at least 3-o(1) [17]. Also, since k controls the variance of the output, at a high level, this result relates to the work of Fain et al. [14] who are not only interested in the expected distortion of rules, but also in the expected squared distortion, essentially forcing randomized rules to have lower variance in their distortion.

Other Related Work The utilitarian analysis of voting rules through the lens of approximation algorithms was first suggested in [9, 10, 26, 27]. Boutilier and Rosenschein [8], Anshelevich et al. [2] were the first to clearly articulate the tension between the objective of maximizing utility (or minimizing cost) and the available information, which is only ordinal; they also termed the resulting gap *distortion*. In the earlier work, such as [9, 10, 26, 27], the focus was on (positive) utilities, and no additional assumptions (such as metric costs) were placed on the utilities.

The role of randomization in reducing distortion has been studied in several prior papers. The fact that randomized voting rules can achieve expected distortion lower than 3 (the known lower bound for deterministic voting rules) was first shown by Anshelevich and Postl [1], who showed that RANDOMDICTATORSHIP achieves expected distortion $3 - \frac{2}{n}$. A slightly improved distortion of $3 - \frac{2}{m}$ was achieved in [20] by randomizing between RANDOMDICTATORSHIP and PROPORTIONALTOSQUARES. This upper bound is best possible among rules that only have access to each candidate's plurality score [19] — in fact, Gross et al. [19] prove a more general lower bound, which establishes that when each voter only communicates her top k < n/2 candidates, every randomized voting rule has distortion at least $3 - \frac{2}{\lfloor n/k \rfloor}$. A lower bound of 2 on the distortion of any randomized voting rule. This conjecture was recently disproved in [12], which established a lower bound of 2.0261 for m = 3 candidates and 2.1126 as the number of candidates $m \to \infty$. Whether any randomized voting rule can achieve expected disortion $3 - \Omega(1)$ for arbitrary m remains an intriguing open question.

Recall that PLURALITYVETO only requires very limited communication from each voter, albeit in an n-round sequential algorithm. In this way, our work relates generally to the study of communication in social choice rules (e.g., [8, 13]), and more specifically to studies of the tradeoff between communication and metric distortion. The recent papers [14, 20] establish related lower bounds: Fain et al. [14] show that any voting rule that only obtains the top k = O(1) candidates of each voter must have squared distortion $\Omega(m)$, in particular implying a bound of $\Omega(m)$ for the distortion of deterministic rules. Kempe [20] proves a slightly more general and stronger lower bound of $\Omega(m/k)$ on the distortion of any deterministic voting rule that only obtains the candidates ranked by each voter in a set K of size k = |K| of positions. Our voting rule avoids these lower bounds by obtaining the bottom candidate from a specified set for each voter; thus, for different voters, the candidates in different positions are queried.

Using randomization, communication can be drastically reduced even compared to our voting rule. Fain et al. [14] present a RANDOMREFEREE mechanism: the mechanism asks two randomly chosen voters for their top choices, and then has a third voter choose between the two proposed candidates. This mechanism, which only requires access to the top choices of two voters plus one bit, achieves not only constant expected distortion, but constant expected *squared* distortion.

Several other recent works have studied the tradeoff between communication and distortion. Mandal et al. [22] study tradeoffs between communication and distortion in the utilitarian model, i.e., without any metric constraints. They also assume that voters actually know their utilities. In this model, they obtain upper and lower bounds on the achievable distortion under communication complexity constraints. Pierczyński and Skowron [25] consider the distortion (and a modified notion of distortion) for approval-based voting (which has reduced communication), in which voters approve all candidates within a certain distance of themselves. They show that under certain parameter settings, for a carefully chosen radius, approval-based voting achieves constant distortion in their sense. Bentert and Skowron [6] consider the approximate implementation of score-based voting rules using low communication. In particular, their techniques in Section 3.2 show that constant distortion 5 + o(1) can be achieved when the number of voters is large, by asking each voter to compare two uniformly random candidates.

2 Preliminaries

Throughout, we use bold face for vectors, and denote the i^{th} component of a vector \mathbf{x} by x_i . Given a set S, let $\Delta(S)$ denote the probability simplex over S, i.e., the set of non-negative weight vectors over S that add up to 1.

An election is a tuple $\mathcal{E} = (V, C, \succ_V)$ consisting of a set of *n* voters *V*, a set of *m* candidates *C* and a ranked-choice profile $\succ_V = (\succ_v)_{v \in V}$; here, \succ_v is the ranking of voter *v*, i.e., a total order over *C*. We say that voter *v* ranks candidate *c* higher than candidate *c'* if $c \succ_v c'$. We also use $c \succcurlyeq_v c'$ when $c \succ_v c'$ or c = c', in which case we say that *v* ranks *c* weakly higher than *c'*.

A voting rule f is an algorithm that returns a candidate $f(\succ_V) \in C$ given a ranked-choice profile \succ_V . We refer to $f(\succ_V)$ as the *winner* of the election \mathcal{E} using the voting rule f, or just as the winner of f if \mathcal{E} is clear from the context. For the most part, we will consider deterministic voting rules; in Section 5, we will also study randomized voting rules.

We refer to the candidate ranked highest by voter v as the *top choice* of v, and denote him by top(v). The candidate ranked lowest by voter v is likewise referred to as the *bottom choice* of v. We use plu(c) to denote the *plurality score* of candidate c, i.e., the number of voters whose top choice is c.

2.1 Metric Distortion

A metric over a set S is a function $d : S \times S \to \mathbb{R}_{\geq 0}$ which satisfies the following conditions for all $a, b, c \in S$: (1) Positive Definiteness: d(a, b) = 0 if and only³ if a = b; (2) Symmetry: d(a, b) = d(b, a); (3) Triangle inequality: $d(a, b) + d(b, c) \geq d(a, c)$.

Given an election $\mathcal{E} = (V, C, \succ_V)$, we say that a metric d over⁴ $V \cup C$ is *consistent* with the ranking \succ_v of voter v if $d(v, c) \leq d(v, c')$ for all $c, c' \in C$ such that $c \succ_v c'$. We say that d is consistent with the ranked-choice profile \succ_V if it is consistent with the ranking \succ_v for all voters $v \in V$. We use $\mathcal{D}(\succ_V)$ to denote the domain of metrics consistent with \succ_V .

The *(utilitarian) social cost* of a candidate c with respect to a metric d is defined as the candidate's sum of distances to all voters: $\cot(c, d) = \sum_{v \in V} d(v, c)$. A candidate c_d^* is *optimal* with respect to the metric d if $c_d^* \in \arg\min_{c \in C} \cot(c, d)$. The *distortion* of a voting rule f, denoted by $\operatorname{dist}(f)$, is the largest possible ratio between the cost of the winner of f and that of an optimal candidate c_d^* , with respect to the worst possible metric $d \in \mathcal{D}(\succ_V)$. That is,

$$\mathsf{dist}(f) = \max_{\succ_V} \sup_{d \in \mathcal{D}(\succ_V)} \frac{\mathsf{cost}(f(\succ_V), d)}{\mathsf{cost}(c_d^*, d)}$$

2.2 Domination Graphs

Domination graphs offer a conceptually simple approach for giving an upper bound of 3 on the distortion of a voting rule. Given an election $\mathcal{E} = (V, C, \succ_V)$, the *domination graph* of a candidate $c \in C$ is the bipartite graph $G^{\mathcal{E}}(c) = (V, V, E_c)$ where $(v, v') \in E_c$ if and only if $c \succeq_v \operatorname{top}(v')$. The main use of these graphs is via the following lemma, due to [24, 21, 17].

Lemma 1. Let f be a voting rule such that for every election $\mathcal{E} = (V, C, \succ_V)$, the domination graph $G^{\mathcal{E}}(f(\succ_V))$ has a perfect matching. Then, f has distortion 3.

Throughout the remainder of the paper, we assume that an election $\mathcal{E} = (V, C, \succ_V)$ is given, and we drop \mathcal{E} from notation when it is clear from the context.

³Our proofs do not require the "only if" condition, so technically, all our results hold for pseudo-metrics, not just metrics.

⁴We only care about the distances between voters and candidates, so d can be defined as a function $d: V \times C \to \mathbb{R}_{\geq 0}$ instead of on $V \cup C$. The triangle inequality can then be written as $0 \leq d(v, c) \leq d(v, c') + d(v', c') + d(v', c)$ for all $v, v' \in V$ and for all $c, c' \in C$.

3 Optimal Distortion via PLURALITYVETO

We now introduce PLURALITYVETO and show that it has distortion 3. PLURALITYVETO requires very limited ordinal information; it only requires one each of the following two types of queries to every voter.

- A top query to a voter v simply returns top(v).
- A *bottom-among query* to a voter v regarding a subset of candidates A returns the bottom choice of v among candidates in A, denoted by $bottom_A(v)$.

Notice that both can of course be easily derived when each voter's full ranking \succ_v is known.

PLURALITYVETO assigns an initial score of plu(c) to each candidate c; doing so only requires making a top query to each voter. These scores are then gradually decreased; when the score of a candidate c reaches zero, we say that c is *eliminated*. Voters are processed one by one in an arbitrary order; this order can be fixed beforehand, or be adaptive and based on voters' preferences. When a voter v is processed, she decrements the score of her bottom choice among the not-yet-eliminated candidates; the bottom choice can be found by making a bottom-among query to v. The winner is the last eliminated candidate. Pseudo-code is given as Algorithm 1.

Algorithm 1 PLURALITY VETO

Input: An election $\mathcal{E} = (V, C, \succ_V)$ **Output:** A winning candidate $c \in C$

- 1: **initialize** score(c) = plu(c) for each $c \in C$
- 2: let (v_1, \ldots, v_n) be an arbitrary ordering of V
- 3: for i = 1, 2, ..., n do

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4: A_i = \{c \in C : \text{score}(c) > 0\}
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- 5: $c_i = \text{bottom}_{A_i}(v_i)$
- 6: **decrement** score (c_i) by 1
- 7: return c_n

Theorem 1. *The distortion of* **PLURALITYVETO** *is 3.*

Proof. We show that $G(c_n)$ has a perfect matching, which proves that PLURALITYVETO has distortion 3 by Lemma 1. Initially, the scores of candidates add up to n, and in each of the n iterations, a positive score is decremented by 1. Thus, the score of each candidate must be 0 at the end. This implies that, for each candidate c, there are plu(c) distinct voters v_i such that $c_i = c$. In other words, for each voter v_i , we can define a unique voter v'_i such that $top(v'_i) = bottom_{A_i}(v_i)$. This means that v_i ranks any candidate in A_i weakly higher than $top(v'_i)$. Since score(c_n) does not get to 0 until the end, note that $c_n \in A_i$ for all i. Hence, each voter v_i ranks c_n weakly higher than $top(v'_i)$, i.e., $(v_i, v'_i) \in E_{c_n}$. Thereby, we have shown that $G(c_n)$ has a perfect matching.

Algorithm 1 is not only quite natural; it also requires only $O(\log m)$ bits of information from each voter v_i , namely, $top(v_i)$ and $bottom_{A_i}(v_i)$. However, implementing the rule with these two queries comes with a trade-off: voters need to wait for possibly n rounds after reporting their top choice.

Note that our proof of Theorem 1 also implies that there is always a candidate whose domination graph has a perfect matching. Indeed, all attempts to resolve the optimal metric distortion conjecture, in one way or another, boiled down to proving the existence of such a candidate. This was done in [17] by giving a stronger existence result, called the ranking-matching lemma. Our rule, as stated so far, does not require this stronger lemma, but in the next section, we show that a simple generalization also implies the general ranking-matching lemma.

4 A Simple Proof of the Ranking-Matching Lemma via FRACTIONALVETO

Our proof of Theorem 1 implied that there always exists a candidate whose domination graph has a perfect matching. The Ranking-Matching Lemma in [17] is a stronger existence result based on an extension of domination graphs in which nodes have arbitrary weights. Given an election $\mathcal{E} = (V, C, \succ_V)$ and weight vectors $\mathbf{p} \in \Delta(V)$ and $\mathbf{q} \in \Delta(C)$, the (\mathbf{p}, \mathbf{q}) -domination graph of a candidate $\tilde{c} \in C$ is the bipartite graph $G_{\mathbf{p},\mathbf{q}}^{\mathcal{E}}(\tilde{c}) = (V, C, E_{\tilde{c}}, \mathbf{p}, \mathbf{q})$; the edge $(v, c) \in E_{\tilde{c}}$ if and only if $\tilde{c} \succeq_v c$. A fractional perfect matching of $G_{\mathbf{p},\mathbf{q}}^{\mathcal{E}}(\tilde{c})$ is a weight function $w : E_{\tilde{c}} \to \mathbb{R}_{\geq 0}$ such that $\sum_{c \in C: (v,c) \in E_{\tilde{c}}} w(v,c) = p_v$ for each voter v and $\sum_{v \in V: (v,c) \in E_{\tilde{c}}} w(v,c) = q_c$ for each candidate c. That is, the total weight of edges incident to each node must be equal to the weight of the node.

A candidate's domination graph has a perfect matching if and only if his $(\mathbf{p}^{uni}, \mathbf{q}^{plu})$ -domination graph has a fractional perfect matching; here, $p_v^{uni} = 1/n$ for all $v \in V$ and $q_c^{plu} = plu(c)/n$ for all $c \in C$. The Ranking-Matching Lemma gives a stronger existence guarantee by asserting that, for any election \mathcal{E} and any $\mathbf{p} \in \Delta(V)$ and $\mathbf{q} \in \Delta(C)$, there is a candidate $c \in C$ whose (\mathbf{p}, \mathbf{q}) -domination graph $G_{\mathbf{p},\mathbf{q}}^{\mathcal{E}}(c)$ has a fractional perfect matching. Gkatzelis et al. [17] give a proof by minimal counterexample for this lemma; thus, their proof is non-constructive. Moreover, it contains elaborate details for defining a minimal counterexample and intricate arguments involving smaller elections where some voters and candidates are removed. This makes the proof somewhat harder to grasp.

We provide a much simpler and constructive proof of the Ranking-Matching Lemma. We generalize PLURALITYVETO in a way that allows any $\mathbf{p} \in \Delta(V)$ and $\mathbf{q} \in \Delta(C)$ to be given as input. We refer to this voting rule as FRACTIONALVETO, and give the pseudo-code as Algorithm 2. Notice that for the special case with weights \mathbf{p}^{uni} and \mathbf{q}^{plu} , FRACTIONALVETO specializes to PLURALITYVETO.

Algorithm 2 FRACTIONALVETO

Input: An election $\mathcal{E} = (V, C, \succ_V)$ along with weight vectors $\mathbf{p} \in \Delta(V)$ and $\mathbf{q} \in \Delta(C)$ **Output:** A candidate $c \in C$ whose (\mathbf{p}, \mathbf{q}) -domination graph has a fractional perfect matching

- 1: **initialize** weight $(v) = p_v$ for each $v \in V$
- 2: **initialize** weight $(c) = q_c$ for each $c \in C$
- 3: while there is a $v \in V$ with weight(v) > 0 do

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4: let v be such a voter
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5: A = \{c \in C : \operatorname{weight}(c) > 0\}
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6: c = bottom_A(v)
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7: \epsilon = \min \{ \operatorname{weight}(v), \operatorname{weight}(c) \}
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8: decrement weight(v) by \epsilon
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9: **decrement** weight(c) by ϵ

10: return c (i.e., the last candidate whose weight is decremented)

Theorem 2. Given any election $\mathcal{E} = (V, C, \succ_V)$, and any weight vectors $\mathbf{p} \in \Delta(V)$ and $\mathbf{q} \in \Delta(C)$, FRACTIONALVETO returns a candidate $c \in C$ whose (\mathbf{p}, \mathbf{q}) -domination graph $G_{\mathbf{p},\mathbf{q}}^{\mathcal{E}}(c)$ has a fractional perfect matching.

Proof. We begin by observing that the **while** loop (lines 3–9) terminates in at most n + m iterations since in each iteration, the weight of either a voter v or a candidate c reaches 0. Next, we show that if a candidate, say \tilde{c} , wins, then $G_{\mathbf{p},\mathbf{q}}(\tilde{c})$ has a fractional perfect matching w.

First, note that, since \tilde{c} is the last candidate whose weight is decremented, $\tilde{c} \in A$ for all iterations. Let us now consider an arbitrary iteration of FRACTIONALVETO in which a voter v is chosen. By definition, vranks any candidate in A weakly higher than the candidate $c = \text{bottom}_A(v)$; in particular, v ranks \tilde{c} weakly higher than c, i.e., $(v, c) \in E_{\tilde{c}}$. Let w assign a weight of ϵ to the edge (v, c), i.e., $w(v, c) = \epsilon$. We now show that w is a fractional perfect matching of $G_{\mathbf{p},\mathbf{q}}(\tilde{c})$.

For each edge $(v, c) \in E_{\tilde{c}}$ to which w assigns positive weight, there exists an iteration where the weights of both v and c are decremented by w(v, c). Hence, the total weight of edges incident to each voter and candidate is equal to how much their weight is decremented until the end. All we need to show is that the weight of each voter v and candidate c reaches 0 at the end as they are initialized, respectively, to p_v and q_c . The total weights of voters and candidates are initially the same, and they stay so after each iteration since only a single voter's and candidate's weights are decremented, and by the same amount. Therefore, when the weight of each voter reaches 0, so does each candidate's. Thus, we have shown that w is a fractional perfect matching of $G_{\mathbf{p},\mathbf{q}}(\tilde{c})$.

5 A Class of Randomized Voting Rules with Distortion 3

As shown in Theorem 1, PLURALITYVETO has distortion 3. Another voting rule that is well known to have distortion 3 (or 3 - 2/n, to be precise) is RANDOMDICTATORSHIP, which returns the top choice of a voter chosen uniformly at random [1]. In the same vein, one can view PLURALITYVETO as choosing a voter *de*-terministically and returning that voter's top choice; this voter is referred to as v'_n in the proof of Theorem 1. This suggests a more general class of randomized voting rules, interpolating between RANDOMDICTATOR-SHIP and PLURALITYVETO: carefully rule out k voters, and return the top choice of a uniformly random voter among the remaining n - k voters. In this section, we introduce such a general rule and show that it achieves distortion at most 3 for every choice of $k \in \{0, 1, ..., n - 1\}$.

A randomized voting rule f is an algorithm which, given a ranked-choice profile \succ_V , returns a probability distribution over candidates $f(\succ_V) \in \Delta(C)$. Writing $\mathbf{w} = f(\succ_V)$, each candidate c is chosen as the winner by f with probability w_c . We refer to \mathbf{w} as the *winner distribution* of f. The cost of a winner distribution under a given metric is the expected cost of the winner, i.e., $\operatorname{cost}(\mathbf{w}, d) = \sum_{c \in C} w_c \cdot \operatorname{cost}(c, d)$. The distortion of a randomized voting rule is still the ratio of the rule's cost to the cost of the optimum solution.

To phrase our generalized voting rule precisely, we observe that a candidate c wins under RANDOM-DICTATORSHIP with probability proportional to his plurality score plu(c). When the score of candidate c_i is decremented at the i^{th} iteration of PLURALITYVETO, suppose that we rule out voter v'_i , as defined in the proof of Theorem 1. Since $top(v'_i) = c_i$, the current score of candidate c_i can be viewed as his plurality score with respect to the remaining voters v'_{i+1}, \ldots, v'_n . Our generalized voting rule (Algorithm 3) therefore simply runs PLURALITYVETO for only k < n iterations, then chooses a candidate with probability proportional to the residual scores. The algorithm is formally given as Algorithm 3.

Algorithm 3 k-ROUNDPLURALITYVETO	
Input: An election $\mathcal{E} = (V, C, \succ_V)$	
Output: A winner distribution $\mathbf{w} \in \Delta(C)$	

1: initialize score(c) = plu(c) for each $c \in C$ 2: let (v_1, \ldots, v_n) be an arbitrary ordering of V3: for $i = 1, 2, \ldots, k$ do 4: $A_i = \{c \in C : score(c) > 0\}$ 5: $c_i = bottom_{A_i}(v_i)$ 6: decrement $score(c_i)$ by 1 7: return $w_c = score(c)/(n-k)$ for all $c \in C$

Our main result in this section is that *k*-ROUNDPLURALITYVETO has distortion at most 3 for any *k*.

Theorem 3. The distortion of k-ROUNDPLURALITYVETO is at most 3 for any $k \in \{0, 1, ..., n-1\}$.

Our proof is based on a generalization of the flow technique from [21] to randomized voting rules. It is encapsulated in Lemma 2 below; this lemma is a straightforward generalization of Lemma 3.1 from [21]. As in that paper, the proof is somewhat technical and long; thus, it is given in the appendix.

The key concept, adopted from [21], is the following flow network. Given an election \mathcal{E} , let $H_{\mathcal{E}} = (V \times C, E)$ be a directed graph with the following edges:

- For every voter v and any pair of candidates c and c' such that $c \succ_v c'$, there is a directed *preference* edge $(v, c) \rightarrow (v, c')$ in E.
- For every candidate c and any pair $v \neq v'$ of distinct voters, there is a (bi-directed) sideways edge $(v, c) \leftrightarrow (v', c)$ in E.

An illustration of a flow network is given in Fig. 1.

For a winner distribution⁵ $\mathbf{w} \in \Delta(C)$ and a candidate c^* , a (\mathbf{w}, c^*) -flow on $H_{\mathcal{E}}$ is a circulation g (i.e., non-negative and conserving flow unless specified otherwise) in which

- For each candidate c and voter v, exactly w_c units of flow originate at the node (v, c).
- Flow is only absorbed at nodes (v, c^*) .

The cost of g at voter v is the total amount of flow absorbed at (v, c^*) , plus the total flow on sideways edges into or out of nodes (v, c), for any candidate c. Formally, $\operatorname{cost}_v(g) = \sum_{e \text{ into } (v, c^*)} g_e + \sum_{c \neq c^*} \sum_{v' \neq v} g_{(v',c) \leftrightarrow (v,c)}$ where $g_{u \leftrightarrow u'} = g_{u \to u'} + g_{u' \to u}$. The cost of g is $\operatorname{cost}(g) = \max_{v \in V} \operatorname{cost}_v(g)$. An illustration of flows and their costs is given in Fig. 2.

The key lemma showing how to use flows to upper-bound the distortion of a voting rule is the following.

Lemma 2. Let f be a randomized voting rule, with the following property: For every election $\mathcal{E} = (V, C, \succ_V)$ and any candidate $c^* \in C$, on the flow network $H_{\mathcal{E}}$, there is a (\mathbf{w}, c^*) -flow g such that $\mathbf{w} = f(\succ_V)$ and $\operatorname{cost}(g) \leq \lambda$. Then, $\operatorname{dist}(f) \leq \lambda$.

⁵Kempe [21] considered only deterministic voting rules; in that case, the distribution \mathbf{w} was restricted to have probability 1 for the deterministic winner, and 0 for all other candidates.

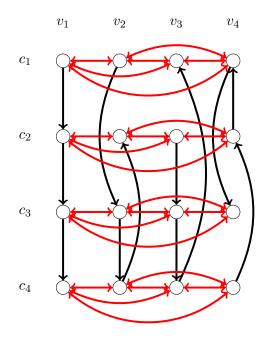


Figure 1: An illustration of a flow network $H_{\mathcal{E}}$. In the example \mathcal{E} , there are 4 voters and 4 candidates. The voter preferences are the following: $v_1 : c_1 \succ c_2 \succ c_3 \succ c_4$; $v_2 : c_1 \succ c_3 \succ c_4 \succ c_2$; $v_3 : c_2 \succ c_3 \succ c_4 \succ c_1$; $v_4 : c_4 \succ c_2 \succ c_1 \succ c_3$. Preference edges are shown in black, while sideways edges are shown in red. For legibility, we have omitted preference edges that could be replaced by a path of two or more other preference edges, e.g., the edge from $(v_1, c_1) \rightarrow (v_1, c_3)$.

We are now ready to give the proof of Theorem 3.

Proof of Theorem 3. Fix an arbitrary candidate $c^* \in C$. We will describe a (\mathbf{w}, c^*) -flow g, and prove that it has cost at most 3. Theorem 3 then follows directly from Lemma 2.

As shown in the proof of Theorem 1, we can sort the voters as v'_1, \ldots, v'_n such that $top(v'_i) = bottom_{A_i}(v_i)$ for all i in the execution of PLURALITYVETO, i.e., without stopping early. Define $V_k = \{v_1, \ldots, v_k\}$ and $\overline{V}_k = \{v_{k+1}, \ldots, v_n\}$; also $V'_k = \{v'_1, \ldots, v'_k\}$ and $\overline{V}'_k = \{v'_{k+1}, \ldots, v'_n\}$. In order to obtain a (\mathbf{w}, c^*) -flow, we must route flow such that for each candidate c and each voter $v \in V$,

In order to obtain a (\mathbf{w}, c^*) -flow, we must route flow such that for each candidate c and each voter $v \in V$, exactly score(c)/(n-k) units of flow originate at the node (v, c). In particular, because score(c) = 0 for candidates $c \notin A_k$, no flow originates at any nodes (v, c) for $c \notin A_k$.

- 1. First, consider a voter $v_i \in V_k$, so that $i \leq k$. Since $A_k \subseteq A_i$, voter v_i ranks all candidates in A_k weakly higher than bottom $_{A_i}(v_i)$. Therefore, g can route all the flow originating at nodes (v_i, c) to $(v_i, bottom_{A_i}(v_i))$ along preference edges. At that point, there is one unit of flow at $(v_i, bottom_{A_i}(v_i))$. This one unit of flow is next routed to $(v'_i, bottom_{A_i}(v_i)) = (v'_i, top(v'_i))$ using a sideways edge. Finally, since $top(v'_i)$ is by definition the top choice of v'_i , the unit can be routed to (v'_i, c^*) using a preference edge, and is then absorbed.
- 2. Next, we (jointly) consider all voters $v_i \in \overline{V}_k$. Fix a candidate $c \in A_k$. Because score(c)/(n-k) units of flow originate at each node (v_i, c) for $v_i \in \overline{V}_k$, and there are $|\overline{V}_k| = n-k$ such nodes, in total, exactly score(c) units of flow originate at these nodes. On the other hand, score(c) is also the number

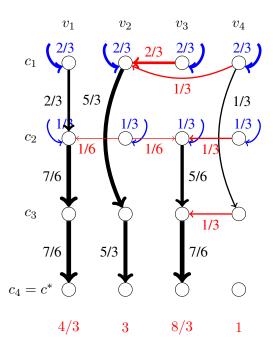


Figure 2: An illustration of a (\mathbf{w}, c^*) -flow on the flow network from Fig. 1, for $\mathbf{w} = (2/3, 1/3, 0, 0)$ and $c^* = c_4$. Edges are only shown when they are used by the flow. Incoming flow is shown in blue. Flow routed along preference edges is shown in black, while flow routed sideways is shown in red to emphasize that it contributes to the cost. The amount of flow is given numerically, and also shown using the width of the lines/arcs. The costs for each voter are shown at the bottom of the corresponding column. The overall cost is the maximum cost, i.e., 3.

of distinct voters $v'_j \in \overline{V}'_k$ such that $top(v'_j) = c$. g distributes all the score(c) units of flow from nodes (v_i, c) (for $v_i \in \overline{V}_k$) to the score(c) nodes (v'_j, c) with $top(v'_j) = c$, in a way that each voter $v'_j \in \overline{V}'_k$ receives one unit of flow. Then, for each voter $v'_j \in \overline{V}'_k$ with $top(v'_j) = c$, the flow g routes the one unit of flow from (v'_j, c) to (v'_j, c^*) using a preference edge; there, it is absorbed. Because flow only originates at nodes (v_i, c) with $c \in A_k$, all the flow is in fact routed to an absorbing node in this way.

In the flow g described above, for each voter v, exactly one unit is sent out on sideways edges, and one unit is received on sideways edges. For voters in V_k , the unit sent is by the first case above; for voters in V'_k , the unit received is by the first case. For voters in \overline{V}_k , the unit sent is by the second case; for voters in \overline{V}'_k , the unit received is by the second case. Finally, each voter v_i absorbs the one received unit of flow at (v_i, c^*) . This proves that cost(g) = 3, completing the proof.

6 Conclusion and Future Work

We showed that a simple deterministic voting rule, called PLURALITYVETO, achieves the optimal metric distortion of 3; the proof is short and simple. We used a generalization of PLURALITYVETO, called FRACTIONALVETO, to prove the Ranking-Matching Lemma of Gkatzelis et al. [17], and we also showed that a class of randomized rules interpolating between RANDOMDICTATORSHIP and PLURALITYVETO all achieve expected distortion at most 3. PLURALITYVETO is a very natural and potentially practical voting rule, and it would be of interest to understand which other properties it satisfies, including the standard axiomatic voting rule properties. One difficulty is that the outcome depends on the order in which the voters are queried in the multi-round veto process. On the other hand, this makes the structure of the set $W \subseteq C$ of potential winners (i.e., candidates who will win for at least one processing order of the voters) an interesting object of study.

Note that W is a subset of candidates whose domination graph has a perfect matching, i.e., potential winners of PLURALITYMATCHING. This has several immediate implications. First, it is possible that for all orderings of voters, the selected candidate fails to be a Condorcet winner, as shown for PLURALITY-MATCHING by Gkatzelis et al. [17]. Second, if the metric space is α -decisive (i.e., for each voter, the ratio between her distance to her top-ranked and second-ranked candidate is at most α [1]) then the distortion bound improves to $2 + \alpha$.

Additional insights can be gleaned from the *Peer Selection* setting, in which the set of voters is the same as the set of candidates, so each voter/candidate ranks herself first; as a result, all such instances are 0-decisive. PLURALITYVETO can be even more easily described in the Peer Selection setting: in each round i, the voter v_i eliminates from consideration the remaining candidate furthest from her.

Here, we consider a slight variant of PLURALITYVETO: the order of voters is chosen adaptively, and the next voter v_{i+1} is always one whose first-place vote was canceled by v_i . In addition, the vote of v_1 is always canceled, so the winner is the top choice of v_n . A proof essentially identical⁶ to the one of Theorem 1 shows that this rule returns a candidate whose domination graph has a perfect matching as well. In turn, we can use this insight to prove that there are at least *two* candidates who can win in Peer Selection, i.e., $|W| \ge 2$. First, if the process is run from an arbitrary v_1 , then some $\hat{v} = v_n$ wins. If the process is run starting from $v_1 = \hat{v}$, the winner must be some other candidate $\tilde{v} \neq \hat{v}$, because \hat{v} is eliminated in the first step. So there are at least two potential winners.

Peer selection also has interesting properties when the voters/candidates are embedded in Euclidean space \mathbb{R}^D . Since each voter, on her turn, eliminates the voter furthest from her, the eliminated voter is always located at a corner of the convex hull of all previously uneliminated ones. In this sense, PLURALITYVETO "peels away" extreme candidates one by one⁷: the convex hull of the voters in A_k (who are not yet eliminated after k rounds) contains no voter from \overline{A}_k . A natural question is whether the set W of potential winners has the same convexity property, i.e., the convex hull of W contains no voter/candidate from $C \setminus W$.⁸ If true, this would show that the potential winners are in a sense "cohesive."

Another compelling direction concerns the incentives under PLURALITYVETO. We described it as a sequential process in which voters are queried about their bottom choice one by one (although of course the process can be fully simulated if each voter's full ranking is known). In describing the sequential process, we assumed that all queries are answered truthfully. While no non-trivial voting rule can be truthful in general [16, 28], truthfulness can be achieved in restricted settings [15]. An interesting direction here is to consider the "Price of Anarchy:" what is the worst distortion of PLURALITYVETO if agents play a subgame perfect equilibrium in the VETO stage instead of truthfully revealing their bottom choice?

⁶Because the vote of v_1 is always canceled, this rule is not technically a special case of PLURALITYVETO.

⁷This behavior also provides some informal intuition for why the final candidate should be close to the geometric median.

⁸For the general case (rather than Peer Selection), this is false: for example, if there is one voter and candidate each on the left and right, but only one candidate (with no voters) in the center, then the left and right candidates can win, but not the center one.

References

- [1] Elliot Anshelevich and John Postl. Randomized social choice functions under metric preferences. In *Proc. 25th Intl. Joint Conf. on Artificial Intelligence*, pages 46–59, 2016.
- [2] Elliot Anshelevich, Onkar Bhardwaj, and John Postl. Approximating optimal social choice under metric preferences. In Proc. 29th AAAI Conf. on Artificial Intelligence, pages 777–783, 2015.
- [3] Elliot Anshelevich, Onkar Bhardwaj, Edith Elkind, John Postl, and Piotr Skowron. Approximating optimal social choice under metric preferences. *Artificial Intelligence*, 264:27–51, 2018.
- [4] Elliot Anshelevich, Aris Filos-Ratsikas, Nisarg Shah, and Alexandros A. Voudouris. Distortion in social choice problems: an annotated reading list. *SIGecom Exchanges*, 19(1):12–14, 2021.
- [5] Elliot Anshelevich, Aris Filos-Ratsikas, Nisarg Shah, and Alexandros A. Voudouris. Distortion in social choice problems: The first 15 years and beyond. In *Proc. 30th Intl. Joint Conf. on Artificial Intelligence*, pages 4294–4301, 2021.
- [6] Matthias Bentert and Piotr Skowron. Comparing election methods where each voter ranks only few candidates. In Proc. 34th AAAI Conf. on Artificial Intelligence, pages 2218–2225, 2020.
- [7] Duncan Black. On the rationale of group decision making. J. Political Economy, 56:23–34, 1948.
- [8] Craig Boutilier and Jeffrey S. Rosenschein. Incomplete information and communication in voting. In Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D. Procaccia, editors, *Handbook* of Computational Social Choice, chapter 10, pages 223–257. Cambridge University Press, 2016.
- [9] Craig Boutilier, Ioannis Caragiannis, Simi Haber, Tyler Lu, Ariel D. Procaccia, and Or Sheffet. Optimal social choice functions: A utilitarian view. *Artificial Intelligence*, 227:190–213, 2015.
- [10] Ioannis Caragiannis and Ariel D. Procaccia. Voting almost maximizes social welfare despite limited communication. Artificial Intelligence, 175(9):1655–1671, 2011.
- [11] Ioannis Caragiannis, Swaprava Nath, Ariel D Procaccia, and Nisarg Shah. Subset selection via implicit utilitarian voting. J. Artif. Intell. Res., 58:123–152, 2017.
- [12] Moses Charikar and Prasanna Ramakrishnan. Metric distortion bounds for randomized social choice. In *Proc. 33rd ACM-SIAM Symp. on Discrete Algorithms*, 2022. to appear. Preprint available at https://arxiv.org/abs/2111.03694.
- [13] Vincent Conitzer and Tuomas Sandholm. Vote elicitation: Complexity and strategy-proofness. In Proc. 17th AAAI Conf. on Artificial Intelligence, pages 392–397, 2002.
- [14] Brandon Fain, Ashish Goel, Kamesh Munagala, and Nina Prabhu. Random dictators with a random referee: Constant sample complexity mechanisms for social choice. In *Proc. 33rd AAAI Conf. on Artificial Intelligence*, pages 1893–1900, 2019.
- [15] Michal Feldman, Amos Fiat, and Iddan Golomb. On voting and facility location. In Proc. 17th ACM Conf. on Economics and Computation, pages 269–286, 2016.

- [16] Alan F. Gibbard. Manipulation of voting schemes: a general result. *Econometrica*, 41(4):587–601, 1973.
- [17] Vasilis Gkatzelis, Daniel Halpern, and Nisarg Shah. Resolving the optimal metric distortion conjecture. In Proc. 61st IEEE Symp. on Foundations of Computer Science, pages 1427–1438, 2020.
- [18] Ashish Goel, Anilesh Kollagunta Krishnaswamy, and Kamesh Munagala. Metric distortion of social choice rules: Lower bounds and fairness properties. In *Proc. 18th ACM Conf. on Economics and Computation*, pages 287–304, 2017.
- [19] Stephen Gross, Elliot Anshelevich, and Lirong Xia. Vote until two of you agree: Mechanisms with small distortion and sample complexity. In Proc. 31st AAAI Conf. on Artificial Intelligence, pages 544–550, 2017.
- [20] David Kempe. Communication, distortion, and randomness in metric voting. In Proc. 34th AAAI Conf. on Artificial Intelligence, pages 2087–2094, 2020.
- [21] David Kempe. An analysis framework for metric voting based on LP duality. In Proc. 34th AAAI Conf. on Artificial Intelligence, pages 2079–2086, 2020.
- [22] Debmalya Mandal, Ariel D. Procaccia, Nisarg Shah, and David P. Woodruff. Efficient and thrifty voting by any means necessary. In *Proc. 33rd Advances in Neural Information Processing Systems*, 2019.
- [23] Hervé Moulin. On strategy-proofness and single peakedness. Public Choice, 35:437–455, 1980.
- [24] Kamesh Munagala and Kangning Wang. Improved metric distortion for deterministic social choice rules. In *Proc. 20th ACM Conf. on Economics and Computation*, pages 245–262, 2019.
- [25] Grzegorz Pierczyński and Piotr Skowron. Approval-based elections and distortion of voting rules. In Proc. 28th Intl. Joint Conf. on Artificial Intelligence, pages 543–549, 2019.
- [26] Ariel D. Procaccia. Can approximation circumvent Gibbard-Satterthwaite? In Proc. 24th AAAI Conf. on Artificial Intelligence, pages 836–841, 2010.
- [27] Ariel D. Procaccia and Jeffrey S. Rosenschein. The distortion of cardinal preferences in voting. In Proc. 10th Intl. Workshop on Cooperative Inform. Agents X, pages 317–331, 2006.
- [28] Mark A. Satterthwaite. Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10:187– 217, 1975.
- [29] Thomas D. Seeley. Honeybee Democracy. Princeton University Press, Princeton, N.J., 2010.

A Proof of Lemma 2

Here, we prove Lemma 2. We restate it for convenience.

Lemma 2 Let f be a randomized voting rule, with the following property: For every election $\mathcal{E} = (V, C, \succ_V)$ and any candidate $c^* \in C$, on the flow network $H_{\mathcal{E}}$, there is a (\mathbf{w}, c^*) -flow g such that $\mathbf{w} = f(\succ_V)$ and $\operatorname{cost}(g) \leq \lambda$. Then, $\operatorname{dist}(f) \leq \lambda$.

Proof. The proof idea is exactly the same as in [21]: we phrase an adversary's optimization problem of maximizing the expected distortion under w as a linear program. By weak duality, any feasible solution to the dual program provides an upper bound on the maximum distortion. We then show that flows directly give rise to such dual-feasible solutions.

The primal linear program is directly adapted from the linear program first given in [3, 18] in the context of a deterministic winner. The variables $x_{v,c}$ of the linear program capture the distances between voters vand candidates c. As such, they must be non-negative and satisfy the triangle inequality; furthermore, they have to be consistent with the voters' preferences \succ_v . The adversary's objective is to maximize the expected distortion, compared to the optimal candidate c^* with knowledge of the metric. (That is, c^* is chosen with hindsight.) Since the distortion is a ratio, to ensure linearity, we phrase the LP as solving the optimization problem of maximizing the *expected cost* under the distribution w, subject to the (optimum) candidate c^* having cost exactly 1. This is equivalent, as any distances can be normalized to ensure this property. The resulting linear program is the following.

$$\begin{array}{lll} \text{Maximize} & \sum_{c} w_{c} \cdot \sum_{v} x_{v,c} \\ \text{subject to} & x_{v,c} \leq x_{v',c} + x_{v',c'} + x_{v,c'} & \text{for all } c, c', v, v' & (\triangle \text{ Inequality}) \\ & x_{v,c} \leq x_{v,c'} & \text{for all } c, c', v \text{ such that } c \succ_{v} c' & (\text{consistency}) & (1) \\ & \sum_{v} x_{v,c^{*}} = 1 & & (\text{normalization}) \\ & x_{v,c} \geq 0 & & \text{for all } c, v. \end{array}$$

After some straightforward rearrangements, the dual linear program is given as LP (2).

$$\begin{array}{lll} \text{Minimize} & \alpha \\ \text{subject to} & \alpha + \sum_{c':c^* \succ_{vc'}} \phi_{c^*,c'}^{(v)} - \sum_{c':c' \succ_{vc^*}} \phi_{c',c^*}^{(v)} \\ & + \sum_{c',v'} \left(\psi_{c^*,c'}^{(v,v')} - \psi_{c',c^*}^{(v,v')} - \psi_{c',c^*}^{(v',v)} \right) \geq w_{c^*} & \text{for all } v \\ & \sum_{c':c \succ_{vc'}} \phi_{c,c'}^{(v)} - \sum_{c':c' \succ_{vc}} \phi_{c',c}^{(v)} \\ & + \sum_{c',v'} \left(\psi_{c,c'}^{(v,v')} - \psi_{c',c}^{(v,v')} - \psi_{c',c'}^{(v',v)} \right) \geq w_c & \text{for all } c \neq c^*, v \\ & \psi_{c,c'}^{(v,v')} \geq 0 & \text{for all } v, v', c, c' \\ & \phi_{c,c'}^{(v)} \geq 0 & \text{for all } v, c, c'. \end{array} \right.$$

The dual has three types of variables:

- $-\psi_{c\,c'}^{(v,v')}$ for the triangle inequality constraints
- $-\phi_{c,c'}^{(v)}$ for the consistency constraints
- α for the normalization constraint.

Because the normalization constraint is an equality constraint, α is unconstrained.

Now consider a candidate c^* for whom the adversary can achieve the largest primal LP value, i.e., who is a witness for the maximum distortion. Fix such a candidate c^* for the rest of the proof, and let $g = g^{c^*}$ be a (\mathbf{w}, c^*) -flow on $H_{\mathcal{E}}$ of cost at most λ — such a flow g exists by assumption of the lemma. We use g to determine values for the dual variables, and show that the proposed dual solution is feasible. Furthermore, we show that the dual objective value is $\cot(g) \leq \lambda$. By weak LP duality, this implies that the primal is upper-bounded by $\cot(g)$, i.e., that the adversary cannot force expected distortion larger than $\cot(g)$ by using the candidate c^* . Since c^* gave the largest distortion bound, we obtain an upper bound on the expected distortion under \mathbf{w} . This will complete the proof. The dual variables are set as follows:

- For each voter v and candidates c, c', we set $\phi_{c,c'}^{(v)}$ to be the flow on the preference edge $(v, c) \to (v, c')$, i.e., we set $\phi_{c,c'}^{(v)} = g_{(v,c)\to(v,c')}$.
- For each pair of voters v, v' and candidate c, we set $\psi_{c,c^*}^{(v,v')}$ to be the flow on the sideways edge $(v,c) \to (v',c)$, i.e., we set $\psi_{c,c^*}^{(v,v')} = g_{(v,c)\to(v',c)}$. Notice that $\psi_{c,c^*}^{(v,v')}$ has four arguments, and we choose c^* for the fourth argument.
- We set $\alpha = \operatorname{cost}(g) = \max_v \operatorname{cost}_v(g)$.
- All other dual variables (in particular $\psi_{c,c'}^{(v,v')}$ for $c' \neq c^*$) are set to 0.

First observe that the dual objective value is indeed $\alpha = \text{cost}(g)$. Also, non-negativity of the dual variables is obviously satisfied. Next, we verify that both sets of dual constraints are satisfied by the dual variables values we assigned.

- To verify the first set of constraints, fix a voter v, and rearrange the constraint to

$$w_{c^{*}} + \left(\sum_{c':c' \succ_{v}c^{*}} \phi_{c',c^{*}}^{(v)} + \sum_{c',v'} \psi_{c^{*},c'}^{(v',v)}\right) - \left(\sum_{c':c^{*} \succ_{v}c'} \phi_{c^{*},c'}^{(v)} + \sum_{c',v'} \psi_{c^{*},c'}^{(v,v')}\right) + \left(\sum_{c',v'} \psi_{c',c^{*}}^{(v,v')} + \sum_{c',v'} \psi_{c',c^{*}}^{(v',v)}\right) \leq \alpha.$$
(3)

We now substitute the definitions of the dual variables. Note that whenever $c' \neq c^*$, our definition implies that $\psi_{c^*,c'}^{(v',v)} = 0$. The left-hand side then becomes

$$w_{c^{*}} + \left(\sum_{c':c' \succ_{v}c^{*}} g_{(v,c') \to (v,c^{*})} + \sum_{v'} g_{(v',c^{*}) \to (v,c^{*})}\right) - \left(\sum_{c':c^{*} \succ_{v}c'} g_{(v,c^{*}) \to (v,c')} + \sum_{v'} g_{(v,c^{*}) \to (v',c^{*})}\right) + \left(\sum_{c',v'} g_{(v,c') \to (v',c')} + \sum_{c',v'} g_{(v',c') \to (v,c')}\right).$$
(4)

Here, notice that the first term is the flow originating at (v, c^*) . The second term is the total flow entering (v, c^*) , and the third term is the total flow leaving (v, c^*) . Thus, the combination of the first three terms is the total flow absorbed at (v, c^*) . The final (fourth) term is the total flow along sideways edges entering or leaving nodes of the form (v, c) for the particular voter v fixed.

Thus, the left-hand side of the (rearranged) dual constraint is exactly the cost $cost_v(g)$. In particular, it is bounded by $cost(g) = \alpha$, implying that the constraint is satisfied by the chosen dual variables.

- Next, we consider the second set of dual constraints, and fix a voter v and candidate $c \neq c^*$. We again rearrange the constraint to make the gist of the analysis clear:

$$w_{c} + \left(\sum_{c':c'\succ_{v}c}\phi_{c',c}^{(v)} + \sum_{c',v'}\psi_{c,c'}^{(v',v)}\right) - \left(\sum_{c':c\succ_{v}c'}\phi_{c,c'}^{(v)} + \sum_{c',v'}\psi_{c,c'}^{(v,v')}\right) + \left(\sum_{c',v'}\psi_{c',c}^{(v,v')} + \sum_{c',v'}\psi_{c',c}^{(v',v)}\right) \le 0.$$

Here, we first notice that because $c \neq c^*$, by definition, all the terms $\psi_{c',c}^{(v,v')}$ and $\psi_{c',c}^{(v',v)}$ in the last two sums are 0. Similarly, in the second and fourth sums, all terms for $c' \neq c^*$ are 0. Substituting the definitions for the remaining dual variables, the left-hand side equals

$$w_c + \left(\sum_{c':c'\succ_v c} g_{(v,c')\to(v,c)} + \sum_{v'} g_{(v',c)\to(v,c)}\right) - \left(\sum_{c':c\succ_v c'} g_{(v,c)\to(v,c')} + \sum_{v'} g_{(v,c)\to(v',c)}\right).$$

Here, observe that the first term w_c is the amount of flow inserted at (v, c), the second term is the amount of flow entering the node (v, c) along preference or sideways edges, and the third term is the amount of flow leaving the node (v, c) along preference or sideways edges. Thus, the left-hand side is exactly the difference between incoming and outgoing flow at (v, c), and because g was assumed to be a valid (\mathbf{w}, c^*) -flow on $H_{\mathcal{E}}$ (and $c \neq c^*$), this net flow must be 0. Therefore, the dual constraint is satisfied.

This completes the proof.